and center at the origin. Since each double point counts as two points the circle cuts the septic curve in 20 points. But this is impossible, hence there is no double point except at the origin. Therefore $\delta=5$. Then from Plücker's equations: $m=32$, $\iota=75$, $\tau=380$. Hence the curve

$$5x^4y-10x^2y^3+y^5+c(x^2+y^2)(x^5-10x^3y+5xy^4)=0,\ c\neq 0.$$

is of class 32, is non-cuspidal, and has five double points at the origin, 75 points of inflection of which five are at the origin, and 380 double tangents. It is obvious that not all the singularities are real.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

330. **Proposed by R. D. CARMICHAEL, Princeton, N. J.**

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n\doteq\infty} x^{1/(2n-1)},\ n=1,\ 2,\ \ldots;\quad f(x)=\lim_{n\doteq\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

No solution of this problem has been received.

331. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Extract the square root of $21+6\sqrt{2}+2\sqrt{21}-6\sqrt{3}-6\sqrt{7}-2\sqrt{6}-2\sqrt{14}$ and also of $4\sqrt{2}+2\sqrt{6}-9-4\sqrt{3}$.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y., and J. SCHEFFER, A.M., Hagerstown, Md.

(a) Assume the root to be of the form

$$a\sqrt{2}+b\sqrt{3}+c\sqrt{7}+d.$$

Squaring and comparing coefficients, we have

$$ab=-1,\ ac=-1,\ ad=3,\ bc=1,\ bd=-3,\ cd=-3,$$
$$2a^2+3b^2+7c^2+d^2=21.$$

Whence $a=-1$, $b=1$, $c=1$, $d=-3$, and the root is $\sqrt{3}-\sqrt{2}+\sqrt{7}-3$.

For second expression, (b), assume root of form $\sqrt{-1}(a+b\sqrt{3}+c\sqrt{2})$. Squaring and comparing coefficients, we have,

$$ab=2,\ ac=-2,\ bc=-1,\ a^2+3b^2+2c^2=9.$$

Whence $a=2$, $b=1$, $c=-1$, and the root is $\sqrt{-1}(2+\sqrt{3}-\sqrt{2})$.

Solved similarly by V. M. Spunar and Levi S. Shively.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

The method used here was suggested to me by Mr. Githens in a solution for cube root. It is considerably simpler than methods ordinarily used.

(a) Let $A=[21+6\sqrt{2}+2\sqrt{21}-(6\sqrt{3}+6\sqrt{7}+2\sqrt{6}+2\sqrt{14})]^{\frac{1}{2}}$
$=\sqrt{(m-n)}$.

Then $\sqrt{[(m-n)(m+n)]}=[157+12\sqrt{2}-(4\sqrt{21}+24\sqrt{42})]^{\frac{1}{2}}=\sqrt{(p-q)}$.

$\sqrt{[(p-q)(p+q)]}=(409-264\sqrt{2})^{\frac{1}{2}}=\sqrt{(r-s)}$.

$\sqrt{[(r-s)(r+s)]}=167$.

Let $\sqrt{(r-s)}=\sqrt{(x+167)}-\sqrt{x}$. Squaring both members and equating rational terms, $2x+167=409$, or $x=121$.

$\therefore \sqrt{(r-s)}=12\sqrt{2}-11$.

Let $\sqrt{(p-q)}=\sqrt{(x+12\sqrt{2}-11)}-\sqrt{x}$.

Then $2x+12\sqrt{2}-11=157+12\sqrt{2}$, or $x=84$.

$\therefore \sqrt{(p-q)}=\sqrt{(73+12\sqrt{2})}-2\sqrt{21}$.

$\sqrt{[(73+12\sqrt{2})(73-12\sqrt{2})]}=71$.

Hence $\sqrt{(73+12\sqrt{2})}=\sqrt{(x+71)}+\sqrt{x}$.

From this, $x=1$ and $\sqrt{(73+12\sqrt{2})}=6\sqrt{2}+1$.

$\sqrt{(p-q)}=6\sqrt{2}+1-2\sqrt{21}$.

Let $\sqrt{(m-n)}=\sqrt{(x+6\sqrt{2}+1-2\sqrt{21})}-\sqrt{x}$.

Then $2x+6\sqrt{2}+1-2\sqrt{21}=21+6\sqrt{2}+2\sqrt{21}$.

$\therefore x=10+2\sqrt{21}$. $\therefore \sqrt{(m-n)}=\sqrt{(11+6\sqrt{2})}-\sqrt{(10+2\sqrt{21})}$.

$\sqrt{[(11+6\sqrt{2})(11-6\sqrt{2})]}=7$, and $\sqrt{(11+6\sqrt{2})}=\sqrt{(x+7)}+\sqrt{x}\ldots(1)$.

$\sqrt{[(10+2\sqrt{21})(10-2\sqrt{21})]}=4$, and $\sqrt{(10+2\sqrt{21})}=\sqrt{(x+4)}+\sqrt{x}\ldots(2)$.

From (1), $x=2$; from (2), $x=3$.

$\therefore \sqrt{(11+6\sqrt{2})}=3+\sqrt{2}$, $\sqrt{(10+2\sqrt{21})}=\sqrt{7}+\sqrt{3}$.

$\therefore \sqrt{(m-n)}=3+\sqrt{2}-\sqrt{7}-\sqrt{3}$.

(b) Let $(-9-4\sqrt{3}+4\sqrt{2}+2\sqrt{6})^{\frac{1}{2}}=\sqrt{(-m+n)}$.

$\sqrt{[(-m+n)(-m-n)]}=\sqrt{[(-m)^2-n^2]}=\sqrt{[73+40\sqrt{3}]}=\sqrt{[p+q]}$.

$\sqrt{[(p+q)(p-q)]}=23$.

Let $\sqrt{(p+q)}=\sqrt{(x+23)}+\sqrt{x}$. Then $2x+23=73$, or $x=25$.

$\therefore \sqrt{(p+q)}=4\sqrt{3}+5$.

Let $\sqrt{(n-m)}=-\sqrt{(x+4\sqrt{3}+5)}+\sqrt{x}$. Then $2x+4\sqrt{3}+5=-9-4\sqrt{3}$, or $x=-7-4\sqrt{3})$.

$\therefore \sqrt{(n-m)}=\sqrt{(-7-4\sqrt{3})}-\sqrt{-2}$.

$\sqrt{[(-7-4\sqrt{3})(-7+4\sqrt{3})]}=1$.

Let $\sqrt{(-7-4\sqrt{3})}=\sqrt{(x+1)}+\sqrt{x}$; then $2x+1=-7$, $x=-4$.

$\therefore \sqrt{(-7-4\sqrt{3})}=\sqrt{(-3)}+2\sqrt{(-1)}$.

$\therefore \sqrt{(n-m)}=\sqrt{(-3)}+2\sqrt{(-1)}-\sqrt{(-2)}$.

Also solved by G. I. Hopkins and A. H. Holmes.

332. Proposed by C. N. SCHMALL, New York City.

Solve the quadratic, $x^2+ax+b=0$, without completing the square.

Solution by ARTEMAS MARTIN, LL. D., Washington, D. C.

Assume $y-\frac{1}{2}a=x$, and substitute in the given quadratic and it becomes

$$(y-\tfrac{1}{2}a)^2+a(y-\tfrac{1}{2}a)+b=0, \text{ or } y^2-\tfrac{1}{4}a^2+b=0;$$

whence $y=\pm\sqrt{(\frac{1}{4}a^2-b)}$ and $x=\pm\sqrt{\frac{1}{4}a^2-b)}-\frac{1}{2}a$.

See *Mathematical Magazine*, Vol. I, No. 9 (January, 1884), p. 146.

Solved similarly by V. M. Spunar , Levi S. Shively and the Proposer.

Professor Hopkins, in his solution, made use of the principle that the sum of the roots is equal to the coefficient of x with sign changed, and the product of the roots is equal to the final term.

S. Lefsehetz sent in solutions of 327 and 328 too late for credit in last issue.

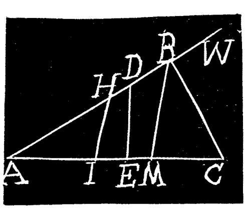
GEOMETRY.

356. Proposed by G. I. HOPKINS, Manchester, N. H.

Required to construct a triangle having given, base, vertical angle, and difference of other two sides.

I. Solution by J. M. ARNOLD, Crompton, R. I.

Let x=the longer side, then $x-d$=the shorter side. Let A=the vertical angle. Then



$$x^2-(x\cos A)^2=(x-d)^2-(b-x\cos A)^2$$

which gives

$$(2b\cos A-2d)x=b^2-d^2.$$

Dividing by 4 and putting in the form of a proportion